2 Linearity, Superposition Principle, and Classification of PDEs

2.1 Linear Equations/Operators

Linearity is a property of differential operators and follows from the basic linearity of differentiation. The property is key to most methodology discussed in this course. For example, from the last section, we considered $L = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. Let c_1, c_2 be arbitrary constants and u_1 and u_2 be arbitrary C^2 functions. Then

$$\begin{aligned} L(c_1u_1 + c_2u_2) &= (c_1u_1 + c_2u_2)_{xx} + (c_1u_1 + c_2u_2)_{yy} \\ &= c_1u_{1xx} + c_2u_{2xx} + c_1u_{1yy} + c_2u_{2yy} \\ &= c_1(u_{1xx} + u_{1yy}) + c_2(u_{2xx} + u_{2yy}) \\ &= c_1L(u_1) + c_2L(u_2). \end{aligned}$$

Thus, L is a linear operator and Laplace's equation is a linear equation.

Definition: An operator $\mathcal{L}$ is **linear** if for any functions $u_1, u_2 \in dom\{\mathcal{L}\}$, and any constants c_1, c_2, $\mathcal{L}(c_1u_1 + c_2u_2) = c_1\mathcal{L}(u_1) + c_2\mathcal{L}(u_2)$.

This property can be checked in two pieces, that is, by showing $\mathcal{L}(u_1 + u_2) = \mathcal{L}(u_1) + \mathcal{L}(u_2)$, then showing $\mathcal{L}(cu) = c\mathcal{L}(u)$. Such a property allows us to "build a solution" from simpler components or gives us the flexibility to satisfy side constraints. Hence, we will restrict the class of problems mostly to linear equations in this course. Of the examples given in Section 1, only the differential operators in (13), and (15) (and the equations in the examples discussing minimal surfaces and halftoning) are nonlinear (not-linear).

Example: Let $L(u) := u_t + cu^2u_x = u_t + \frac{c}{3}(u^3)_x$. Then note that
$L(ku) = ku_t + \frac{c}{3}k^3(u^3)_x \neq kL(u) = ku_t + \frac{c}{3}k(u^3)_x$.

Remark: Linearity is a property of the PDE *operator*, though we speak about an equation being linear. For example, is $xu_{xy} + 4y^2u = \sin(x)$ a linear equation? It is, but here is how we arrive at that. Let $L := x\frac{\partial^2}{\partial x \partial y} + 4y^2$. Then you can show that L satisfies the above definition, that is, $L(c_1u_1 + c_2u_2) = c_1L(u_1) + c_2L(u_2)$. The (non-homogeneous) terms that do not involve the unknown, or any of its derivatives, is not material to the definition of linearity.

2.2 First-order Equations in Two Independent Variables

If the two variables are represented by x and y, then we have the general form

$$F(x, y, u(x, y), \frac{\partial u}{\partial x}(x, y), \frac{\partial u}{\partial y}(x, y)) = 0.$$

Such general first-order PDEs are complicated to solve, and often do not have a solution. An actual example of this case is

$$(\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2 + 1 = 0.$$

A first order **linear** PDE in two variables has the form

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} + cu = 0$$

where the coefficients a, b, c depend, at most, on x and y (but not on u or its derivatives). While equation (15) in Section 1 of these Notes is *not* linear, $\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$ is linear. In the section on first-order PDEs we will further classify these equations based on structure. Of course one can discuss PDEs (both first-order and higher order) with more than two independent variables, but we will rarely do so since every technique we discuss and the physical problems mentioned can be discussed for functions of two variables without the burden of the extra abstract notation required by the more general case.

A second-order **linear** PDE in two independent variables (again call them x and y) has the form

$$A\frac{\partial^2 u}{\partial x^2} + 2B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu + G = 0. \tag{1}$$

where A, B, C, D, E, F, G are functions that *depend at most on* x, y. (Writing $2B$ here instead of B is a convenience for classification purposes; see below.)

Examples: From Section 1, for the one-dimensional wave equation (2), we can consider $A = c^2$, $C = -1$ (identifying y with t), and $B = D = E = F =$

$G = 0$. A one space dimensional model from the study of gas dynamics is Burger's equation,

$$\epsilon \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} - u\frac{\partial u}{\partial x} = 0$$

Thus, $A = \epsilon$, $D = -u$, $E = -1$, with the rest of the coefficients being zero. But Burger's equation is nonlinear because D depends on the dependent variable u.
Remark: In (1) the unknowns are $u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}$, so (1) represents a summation of terms with at most one unknown in each term, and that term is proportional to the unknown. That is, as a linear equation, (1) represents a "monomial" in at most 6 unknowns.

Exercises: Which of the following equations are linear[1]:

1. $u_{xx} + x^2 u_x - u_{yy} = y^3$
2. $u_{xx} - uu_x + u_{yy} = 0$
3. $\sin(xy)u_{xx} - \cos(xy)u_{xy} = 0$
4. $u_y u_x - u_{yy} = 0$

Homogeneous versus Nonhomogeneous Equations
In (1) the term G represents all such terms in the equation that do not depend on u or any of its derivatives. If, in the domain where the equation holds, $G \equiv 0$, the equation is **homogeneous**. Otherwise, if G is not zero somewhere in its domain, the equation is **nonhomogeneous**.

Remark: When checking an equation for the linearity property, the definition considers the operator without the G term. For example,

$$u_t + \nu u_x = D u_{xx} + a e^{-(x-x_0)^2}, \text{ where } x_0, a, \nu \text{ are constants}$$

represents a linear equation which is nonhomogeneous because of the $ae^{-(x-x_0)^2}$ term. The operator $\mathcal{L} = \frac{\partial}{\partial t} + \nu \frac{\partial}{\partial x} - D\frac{\partial^2}{\partial x^2}$ satisfies the definition of linear operator. The νu_x term represents an advection term, a kind of transport mechanism. Think of u representing a concentration of a pollutant in the air (or river) being diffused throughout the environment (resp. the river) with

[1] Answers: equations 1 and 3 are linear, and 2 and 4 are nonlinear. Why?

diffusion coefficient D, and being convected due to a steady wind (or current). The nonhomogeneity $ae^{-(x-x_0)^2}$ is considered a source term for the pollutant.

Remark: Recall from your ODE course that you derive a fundamental set of solutions for a homogeneous equation, then use that knowledge in a variation-of-parameters method, or undetermined-coefficients method to construct a particular solution to the nonhomogeneous equation. That is, to solve a nonhomogeneous equation, you use knowledge gained from solving the (reduced) homogenous equation. We will take the same basic approach in PDEs, first solving homogenous equations, then using knowledge of the form to attack nonhomogeneous equations. This will be particularly evident when I discuss boundary value problems (spatial domains are bounded).

2.3 Superposition Principle and the Subtraction Principle

If $u_1, u_2, \ldots, u_N$ are solutions to the *homogeneous linear* PDE $\mathcal{L}(u) = 0$, then so is the linear combination $u = c_1u_1 + c_2u_2 + \ldots + c_Nu_N$, where the $c_j's$ are arbitrary constants. This is just a consequence of the linearity property of $\mathcal{L}$:
$\mathcal{L}(c_1u_1 + c_2u_2 + \ldots + c_Nu_N) = c_1\mathcal{L}(u_1) + c_2\mathcal{L}(u_2) + \ldots + c_N\mathcal{L}(u_N) = 0 + 0 + \ldots + 0 = 0.$

This is the codification of the **Superposition principle**.

Remark: This statement does not apply to nonhomogenous equations. For example, suppose u_1, u_2 are two solutions to $u_{xx} + u_{yy} = 1$. then $u = u_1 + u_2$ is a solution to $u_{xx} + u_{yy} = 2$, not $u_{xx} + u_{yy} = 1$.

For the *Subtraction Principle*, if u_1 and u_2 are two solutions to a **nonhomogeneous linear** equation, the $u = u_1 - u_2$ is a solution to the associated homogeneous equation. (By linearity of the operator, if $\mathcal{L}(u_1) = f$ and $\mathcal{L}(u_2) = f$, then $\mathcal{L}(u_1 - u_2) = \mathcal{L}(u_1) - \mathcal{L}(u_2) = f - f = 0$.)

2.4 Classification of 2nd order, Linear, PDEs in Two Independent Variables

Classification of equation (1) concerns just the **principle part of the PDE operator**, namely $A\partial^2/\partial x^2 + 2B\partial^2/\partial x\partial y + C\partial^2/\partial y^2$, so the classification below also applies to nonlinear equations of the form

$$A\frac{\partial^2 u}{\partial x^2} + 2B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + R(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) = 0 \ .$$

The principle part of the PDE operator drives the main characteristics of solutions to the equation.
Definition: The **discriminant** is $\mathcal{D} := B^2 - AC$. If

1. $\mathcal{D} > 0$, then equation (1) is **hyperbolic**;
2. $\mathcal{D} < 0$, then equation (1) is **elliptic**;
3. $\mathcal{D} = 0$, then equation (1) is **parabolic**

Remarks
Some texts will write the principle part of the operator with a B instead of $2B$; in this case the discriminant is defined as $\mathcal{D} := B^2 - 4AC$. Then the classification again follows. But do NOT mix up the operator notation and the discriminant definition.

We will be driven by this classification. So the first part of the course really handles equations in each category separately. This is quite different than the case of 2nd order linear ODEs.

The PDE coefficients A, B, C can be functions of x and y, so the definition of hyperbolic, elliptic, parabolic is *local*; that is, as (x, y) ranges over the equations domain, the equation may change type. Such equations are change-of-type equations. An example of this is **Tricomi's equation** $u_{xx} - xu_{yy} = 0$, which was originally introduced to try to understand linear transonic flow better. (The linearized equations for transonic flow, that is flow past an object moving near the speed of sound, is change-of-type, but harder to analyze than Tricomi's equation.) Note in this case that $\mathcal{D} = x$, so the equation is hyperbolic for $x > 0$, elliptic for $x < 0$, and $x = 0$ is a parabolic line for the equation.

Why the classification? Because solution behavior has qualitative similarity for equations pulled out of the same "bucket", but solution behavior is quite different between equations pulled from different buckets (if we consider the equations falling into one of the three buckets: the hyperbolic bucket, the parabolic bucket, and the elliptic bucket). For example, discontinuities in boundary data for the wave equation (2) in Section 1, or the one-space version (9), will be propagated throughout the domain along paths called characteristics. So, the "smoothness" of solutions of these hyperbolic equations will be no more so than the boundary data. However, the heat equation, or Black-Scholes equation (11), which come from the parabolic bucket, will have solutions that are infinitely differentiable independent of any non-smoothness of the boundary data. Put another way, such "boundary information" like singularities, is lost immediately in time due to the diffusion process inherent in parabolic equations.

Examples: $x^2 u_{xx} + 2xu_{xt} + u_{tt} = u_t$ has $A = x^2$, $B = x$, $C = 1$, so $\mathcal{D} = x^2 - x^2 = 0$; so the equation is parabolic everywhere. However, $yu_{xx} - 2u_{xy} + xu_{yy} = 0$ has $A = y$, $B = -1$, and $C = x$, so $\mathcal{D} = 1 - xy$. In this case the equation is parabolic on the hyperbola $xy = 1$, elliptic in the two disconnected convex regions $\{xy > 1\}$, and hyperbolic in the connected region $\{xy < 1\}$.

Exercises: Classify the following PDEs; that is, what region is the equation elliptic, hyperbolic, or parabolic?[2]

1. $xu_{xx} - 4u_{xt} = 0$
2. $u_{xx} - 6u_{xy} + 12u_{yy} = 0$
3. $x^2 u_{xx} - y^2 u_{yy} = 0$
4. $\sin(xy) u_{xy} = 0$
5. $(1 + x)u_{xx} + 2xyu_{xy} - y^2 u_{yy} = 0$

[2]Equation 1 is hyperbolic everywhere, and equation 2 is elliptic everywhere; equation 3 is parabolic on the principle axes, and hyperbolic elsewhere; equation 4 is hyperbolic except where $\sin(xy) = 0$; equation 5 is parabolic on the x-axis, and hyperbolic elsewhere.

2.5 Comment on what will and will not be discussed in these Notes

The main equations to be solved under various circumstances are the heat, wave, and Laplace's equations, which represent prototypical equations from the parabolic, hyperbolic, and elliptic "buckets". We will briefly discuss Poisson's equation (nonhomogeneous Laplace's equation), the telegrapher's equation, and the beam equation. The telegrapher's equation is a wave equation with lower order terms that allow us to introduce concepts like dissipation and dispersion, which are important concepts in a variety of physical circumstances. The beam equation, or more precisely the Euler-Bernoulli beam equation, is a classical higher-order equation that is still physically important and worth being introduced to. We will show that our methods which we developed for second-order equations can be applied to this fourth-order equation. We will also take a little time to solve some classes of first-order equations since there are interesting physical problems that lead to such equations (like (15) in Section 1).

We will not have time to delve into systems of PDEs. We will not discuss equations like the Korteweg-deVries equation (13) of Section 1, or the eikonal equation

$$(u_x)^2 + (u_y)^2 = c^2$$

or the sine-Gordon equation

$$u_{tt} - c^2\nabla^2 u + \sin(u) = 0$$

because they are nonlinear. They are interesting equations in their own right, but techniques for studying them go beyond this course. We will also not study problems like (12) of Section 1 because all parameters in our equations in this course will be considered real numbers.

As previously mentioned we will first study problems that are defined, spatially, on the real line (no finite spatial boundaries) to characterize some important differences in solution behavior between parabolic and hyperbolic equations. Then we will add a boundary to see how boundaries can reflect or absorb information from the solution. Finally, we will examine problems on bounded domains most relevant to various physical situations.

Below are a few exercises I recommend you go through.

1. Define the following:

(a) order of a pde
(b) superposition principle
(c) classical solution
(d) elliptic pde

2. Which of the following operators are linear:

   (a) $\mathcal{L}u = u_x + xu_y$
   (b) $\mathcal{L}u = u_x + uu_y$
   (c) $\mathcal{L}u = u_x + u_y^2$
   (d) $\mathcal{L}u = \sqrt{1+x^2}\ \cos(y)u_x + u_{yxy} - \tan^{-1}(x/y)u$

3. For the following equations, give the order, and state whether each is linear or nonlinear. If it is linear, indicate whether it is homogeneous or nonhomogeneous.

   (a) $u_t - u_{xx} + 1 = 0$
   (b) $u_t - u_{xx} + xu = 0$
   (c) $u_t - u_{xxt} + uu_x = 0$
   (d) $u_{tt} - u_{xx} + x^2 = 0$
   (e) $iu_t - u_{xx} + u/x = 0$
   (f) $u_x + e^y u_y = 0$
   (g) $u_t - u_{xxxx} + \sqrt{1+u} = 0$

4. For the following, show the given function solves the given equation:

   (a) $u(x,y) = \ln(\sqrt{x^2+y^2})$, $u_{xx} + u_{yy} = 0$
   (b) $u(x,y) = f(x)g(y)$, $uu_{xy} = u_x u_y$, where f, g are arbitrary differentiable functions of one variable.
   (c) $u(x,t) = \frac{1}{2\sqrt{\pi Dt}} e^{-x^2/4Dt}$, $u_t - Du_{xx} = 0$ ($D > 0$ is a constant)
   (d) $u(x,t) = x^2 + 2t$, $u_t - u_{xx} = 0$
   (e) $u(x,t) = e^{-3t}\sin(x)$, $u_t - 3u_{xx} = 0$

5. For what values of a and b will the following functions $u(x,y)$ solve the given pde?

(a) $u(x, y) = e^{at} \sin(bx)$, for $u_t - Du_{xx} = 0$

(b) $u(x, y) = bx + f(e^{-ax}y)$, for $u_x - yu_y = 0$, where f is an arbitrary differentiable function.